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On the Greatest Zero of an Orthogonal Polynomial*

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1. CHEBYSHEV'S MAXIMUM PRINCIPLE AND ITS EXTENSIONS

Let w be a nonnegative weight function on $(-\infty, \infty)$ for which $x^n w(x) \in L^1$ $(n = 0, 1, \ldots)$. We construct the sequence of orthonormal polynomials $p_n(w; x) = \gamma_n(w) x^n + \cdots$ $(\gamma_n(w) > 0)$ satisfying

$$\int_{-\infty}^{\infty} p_m(w; x)\, p_n(w; x)\, w(x)\, dx = \delta_{mn}. \tag{1}$$

In what follows we assume that w is even. Then the polynomials p_n satisfy the recurrence relation

$$xp_n(w; x) = c_{n+1/2}(w)\, p_{n+1}(w; x) + c_{n-1/2}(w)\, p_{n-1}(w; x). \tag{2}$$

It is well known that all zeros x_{kn} of $p_n(w)$ are real and simple. Let us denote by $X_n(w)$ the greatest zero of $p_n(w)$. The sequence $\{X_n(w)\}$ is increasing and, by virtue of a result of Chebyshev,

$$X_n(w) = \max_{\pi_{n-1}} \frac{\int_{-\infty}^{\infty} x\pi_{n-1}^2(x)\, w(x)\, dx}{\int_{-\infty}^{\infty} \pi_{n-1}^2(x)\, w(x)\, dx} \qquad (n = 1, 2, \ldots); \tag{3}$$

the maximum being attained for $\pi_{n-1}(x) = p_n(w; x)/(x - X_n(w))$. Here and in what follows π_k denotes an arbitrary real polynomial $\neq 0$ of degree $\leqslant k$. Relation (3) is valid for arbitrary weights.

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THEOREM 1. *For even weights,* (3) *can be extended to*

$$X_n^2(w) = \max_{\pi_{n-2}} \frac{\int_{-\infty}^{\infty} x^2 \pi_{n-2}^2(x)\, w(x)\, dx}{\int_{-\infty}^{\infty} \pi_{n-2}^2(x)\, w(x)\, dx} \qquad (n = 2, 3, \ldots), \tag{4}$$

$$X_{2\nu-1}^4(w) \geqslant \max_{\pi_{2\nu-1}} \frac{\int_{-\infty}^{\infty} x^4 \pi_{2\nu-4}^2(x)\, w(x)\, dx}{\int_{-\infty}^{\infty} \pi_{2\nu-4}^2(x)\, w(x)\, dx} \qquad (\nu = 2, 3, \ldots) \tag{5}$$

and

$$X_{2\nu-1}^3(w) \geqslant \max_{\pi_{2\nu-1}} \frac{\int_{-\infty}^{\infty} x^3 \pi_{2\nu-3}^2(x)\, w(x)\, dx}{\int_{-\infty}^{\infty} \pi_{2\nu-3}^2(x)\, w(x)\, dx} \qquad (\nu = 2, 3, \ldots). \tag{6}$$

Relations (4), (5), and (6) are, along with (3), obvious consequences of the quadrature formula

$$\int_{-\infty}^{\infty} \pi_{2n-1}(x)\, w(x)\, dx = \sum_{k=1}^{n} \lambda_{kn}(w)\, \pi_{2n-1}(x_{kn}) \tag{7}$$

and the fact that all Christoffel numbers $\lambda_{kn}(w)$ are positive.

2. DEPENDENCE OF THE GREATEST ZERO ON THE RECURSION COEFFICIENTS

Let us set in (3)

$$\pi_{n-1}(x) = \sum_{k=0}^{n-1} J_k\, p_k(w; x). \tag{8}$$

It follows from (2) that

$$c_{k+1/2}(w) = \int_{-\infty}^{\infty} p_k(w; x)\, p_{k+1}(w; x)\, w(x)\, dx = \frac{\gamma_k(w)}{\gamma_{k+1}(w)} > 0. \tag{9}$$

Inserting (8) in (3) and considering (1), (9) we obtain

THEOREM 2. *We have*

$$X_n(w) = \max_{J_k \geqslant 0} 2\, \frac{\sum_{k=0}^{n-2} c_{k+1/2} J_k J_{k+1}}{\sum_{k=0}^{n-1} J_k^2}. \tag{10}$$

THEOREM 3. *Let the recursion coefficients belonging to the weights* w_1 *and* w_2 *be denoted by* $\{c_{k+1/2}^{(1)}\}$ *and* $\{c_{k+1/2}^{(2)}\}$, *respectively, and let*

$$c_{k+1/2}^{(1)} \leqslant A_n c_{k+1/2}^{(2)} \qquad (k = 0, 1, \ldots, n-2).$$

Then we have

$$X_n(w_1) \leqslant A_n X_n(w_2). \tag{11}$$

As a first application of Theorem 3 we show

THEOREM 4. *We have*

$$X_n(w) \leqslant 2 \cos \frac{\pi}{n+1} \max_{k \leqslant n-2} c_{k+1/2}. \tag{12}$$

This theorem improves a previous result of the author [1] by the factor $\cos \pi/(n+1)$. To prove Theorem 4, let us apply Theorem 3 with $w_1 = w$ (i.e., $c^{(1)}_{k+1/2} = c_{k+1/2}$), $A_n = 2 \max_{k \leqslant n-2} c_{k+1/2}$ and $c^{(2)}_{k+1/2} = \frac{1}{2}$. The orthogonal polynomials belonging to the recursion formula (2) with $c_{k+1/2} = \frac{1}{2}$ are, apart of the normalization factor, the Chebyshev polynomials of second kind $U_n(x)$, defined by

$$U_n(\cos \theta) = \frac{\sin(n+1)\,\theta}{\sin \theta}.$$

The greatest zero of $U_n(x)$ is $X_n(w_2) = \cos \pi/(n+1)$. Inserting this value in (11) we obtain (12).

In [1] we observed that (10) implies

$$X_n(w) \geqslant \max_{k \leqslant n-2} c_{k+1/2}. \tag{13}$$

THEOREM 5. *We have*

$$X^4_{2\nu}(w) > X^4_{2\nu-1}(w) \geqslant \max_{k \leqslant 2\nu-4} \{c^2_{k+3/2}(w)\, c^2_{k+1/2}(w) \tag{14}$$

$$+ [c^2_{k+1/2}(w) + c^2_{k-1/2}(w)]^2 + c^2_{k-1/2}(w)\, c^2_{k-3/2}(w)\} \qquad (\nu \geqslant 3).$$

To compare the estimates (13) and (14) let us assume that the $c_{k+1/2}$'s are slowly varying. Then the value of the expression on right of (14) is asymptotically $6(\max c_{k+1/2})^4$. Consequently (14) improves (13) asymptotically by a factor $\sqrt[4]{6} = 1.565\ldots$.

To prove (14), let us insert in (5) $\pi_{2\nu-4}(x) = p_k(w; x)$. By iterated application of (2)

$$x^2 p_k(w; x) = c_{k+3/2} c_{k+1/2}\, p_{k+2}(w; x) + (c^2_{k+1/2} + c^2_{k-1/2})\, p_k(w; x)$$

$$+ c_{k-1/2} c_{k-3/2}\, p_{k-2}(w; x) \qquad (k \geqslant 2) \tag{15}$$

and consequently

$$\int [x^2 p_k(w; x)]^2\, w(x)\, dx = c^2_{k+3/2} c^2_{k+1/2} + (c^2_{k+1/2} + c^2_{k-1/2})^2 + c^2_{k-1/2} c^2_{k-3/2} \tag{16}$$

which proves (14).

The main result of this chapter is

THEOREM 6. *If the recursion coefficients satisfy*

$$\lim_{k\to\infty} \frac{c_{k+1/2}(w)}{c_{k-1/2}(w)} = 1 \tag{17}$$

then we have

$$\liminf_{n\to\infty} \frac{X_n(w)}{c_{n-1/2}(w)} \geqslant 2. \tag{18}$$

Proof. Let m be an arbitrary but fixed natural integer. Let $\varepsilon > 0$ be fixed and $N = N(m, \varepsilon)$ be chosen so that

$$c_{n-1/2-r}(w) > (1-\varepsilon)\, c_{n-1/2}(w) \qquad (n \geqslant N;\ r = 1, 2, \ldots, m) \tag{19}$$

Inserting $J_0 = J_1 = \cdots = J_{n-m-1} = 0$ in (10) and taking (19) in consideration, we find that

$$X_n(w) \geqslant (1-\varepsilon)\, c_{n-1/2}(w) \max_{J_k \geqslant 0} 2 \left(\sum_{k=n-m}^{n-2} \tfrac{1}{2} J_k J_{k+1} \Big/ \sum_{k=n-m}^{n-1} J_k^2 \right).$$

By Theorem 2 the maximum expression on the right equals to the greatest zero $\cos \pi/(m+1)$ of $U_m(X)$ (see the proof of Theorem 4). Consequently we have for arbitrary ε and m and for sufficiently large values of n

$$X_n(w) \geqslant 2(1-\varepsilon) \cos \frac{\pi}{m+1}\, c_{n-1/2}(w)$$

which proves (18).

THEOREM 7. *If the recursion coefficients satisfy* $\lim_{k\to\infty} c_{k+1/2}(w) = \infty$ *and* (17) *then*

$$\lim_{n\to\infty} \frac{X_n(w)}{\max_{k \leqslant n} c_{k-1/2}} \leqslant 2. \tag{20}$$

Proof. It follows from (12) and (17) that

$$\limsup_{n\to\infty} \frac{X_n(w)}{\max_{k\leqslant n} c_{k-1/2}} \leqslant 2. \tag{21}$$

Now let $\{n_r\}$ be the increasing sequence of integers for which

$$\max_{k\leqslant n_r} c_{k-1/2} = c_{n_r-1/2}.$$

By virtue of Theorem 6 we have

$$\liminf_{r\to\infty} \frac{X_{n_r}(w)}{c_{n_r-1/2}} \geqslant 2. \tag{22}$$

For an arbitrary n we can find r such that

$$n_r \leqslant n < n_{r+1}, \tag{23}$$

and consequently

$$c_{n-1/2}(w) \leqslant c_{n_r-1/2}(w). \tag{24}$$

Note that, in consequence of $c_{n-1/2}(w) \to \infty$, $n \to \infty$ implies $n_r \to \infty$. Since the sequence $\{X_n(w)\}$ is increasing, clearly

$$X_n(w) \geqslant X_{n_r}(w). \tag{25}$$

By (22), (24), and (25)

$$\liminf_{n\to\infty} \frac{X_n(w)}{\max_{k\leqslant n} c_{k-1/2}(w)} \geqslant \liminf_{n\to\infty} \frac{X_{n_r}(w)}{c_{n_r-1/2}} \geqslant 2; \tag{26}$$

(21) and (26) together imply (20).

3. Applications

A

F. Pollaczek [5] proved that the sequence $P_n^{(\lambda)}(x)$ of polynomials defined by the recurrence relation

$$xP_n^{(\lambda)}(x) = \frac{n+1}{2} P_{n+1}^{(\lambda)}(x) + \frac{n-1+2\lambda}{2} P_{n-1}^{(\lambda)}(x) \tag{27}$$

and $P_0^{(\lambda)}(x)=1$, $P_1^{(\lambda)}(x)=2x$ is orthogonal with respect to the weight

$$w^{\lambda}(x)=|\Gamma(\lambda+ix)|^2, \qquad -\infty<x<\infty. \tag{28}$$

To find the coefficients in the recursion formula for the orthonormal polynomials, we apply the following

THEOREM 8. *Let $\{R_n(x)\}$ be a sequence of orthogonal polynomials which satisfy the recurrence relation*

$$xR_n(x)=A_nR_{n+1}(x)+B_nR_n(x)+C_nR_{n-1}(x). \tag{29}$$

Then the coefficients in the recurrence relation for the orthonormal polynomials are

$$c_{n-1/2}=\sqrt{A_{n-1}C_n}. \tag{30}$$

By virtue of Theorem 8 we infer from (27) that

$$c_{n-1/2}(w^{\lambda})=\tfrac{1}{2}\sqrt{n(n+2\lambda-1)}.$$

Let us insert in (10) $J_k=O$ $(0\leqslant k\leqslant n-m-1)$ and $J_{n-l-1}=\eta_l$ $(0\leqslant l\leqslant m-1)$. It follows for every $m<n-1$

$$X_n(w)\geqslant 2\min_{n-m\leqslant k<n-1} c_{k-1/2}\max 2\left(\sum_{l=0}^{m-2}\tfrac{1}{2}\eta_l\eta_{l+1}\Big/\sum_{l=0}^{m-1}\eta_l^2\right).$$

The maximum expression above is the greatest zero of $U_m(x)$, i.e., $\cos\pi/(m+1)$ so that

$$X_n(w)\geqslant 2\min_{n-m\leqslant k<n-1} c_{k-1/2}(w)\cdot\cos\frac{\pi}{m+1}. \tag{31}$$

Applying this relation to $w=w^{\lambda}$, $c_{k-1/2}(w_{\lambda})=\frac{1}{2}\sqrt{k(k+2\lambda-1)}$ we have

$$\begin{aligned} X_n(w^{\lambda})&\geqslant\sqrt{(n-m)(n-m+2\lambda-1)}\cos\frac{\pi}{m+1}\\ &=\sqrt{n(n+2\lambda-1)}\left[1-O\left(\frac{m}{n}\right)\right]\left[1-O\left(\frac{1}{m^2}\right)\right]. \end{aligned}$$

With the choice $m=[n^{1/3}]$ we finally obtain

$$\sqrt{n(n+2\lambda-1)}[1-O(n^{-2/3})]\leqslant X_n(w^{\lambda})\leqslant\sqrt{n(n+2\lambda-1)}\cos\frac{\pi}{n+1}. \tag{32}$$

The second half of (32) is obtained from (12).

B

A similar argument can be applied to Hermite polynomials $H_n(x)$. From the recursion formula

$$xH_n(x)=\tfrac{1}{2}H_{n+1}(x)+nH_{n-1}(x)$$

we obtain by Theorem 8 that $c_{n-1/2}=\sqrt{n/2}$. The argument used in part A gives the inequalities for the greatest zero X_n of $H_n(x)$

$$\sqrt{2n}-O(n^{-1/6})<X_n<\sqrt{2n}\cos\frac{\pi}{n+1}. \tag{33}$$

C

Let $w_{\rho m}(x)=|x|^\rho \exp\{-|x|^m\}$. We proved in [4] that

$$\lim_{n\to\infty} n^{-1/m}c_{n-1/2}(w_{\rho m})=\left[\frac{\Gamma(m+1)}{\Gamma(m/2)\,\Gamma((m/2)+1)}\right]^{-1/m} \tag{34}$$

is valid for $m=2, 4, 6$, $\rho>-1$ and we conjectured that (34) holds for every $m>0$. It follows from Theorem 7 that

$$\lim_{n\to\infty} n^{-1/m}X_n(w_{\rho m})=2\left[\frac{\Gamma(m+1)}{\Gamma(m/2)\,\Gamma((m/2)+1)}\right]^{-1/m} \tag{35}$$

is valid for $m=2, 4, 6$ and every $\rho>-1$. Extending our earlier conjecture, we expect that (35) is true for every $m>0$ and $\rho>-1$.

4. Inequalities for the Greatest Zero

Let

$$w_Q(x)=\exp\{-2Q(x)\}, \tag{36}$$

where $Q(x)$ is an even differentiable function. In our paper [3] we proved that if $x^sQ'(x)$ is increasing in $(0<x<\infty)$ for some $s<1$ then

$$c_1q_n<X_n(w_Q)<c_2q_n \tag{37}$$

holds for certain positive numbers c_1 and c_2. Here q_n is the positive solution of the equation

$$q_nQ'(q_n)=n. \tag{38}$$

Note the q_n tends increasingly to ∞ for $n \to \infty$. We make now the stronger assumption that $Q(x)$ is a nonconstant convex function and give numerical values for c_1 and c_2 in (37).

LEMMA 1. *We have*

$$c_{n-1/2}(w_Q) = \int_{-\infty}^{\infty} x p_n(w_Q; x)\, p_{n-1}(w_Q; x)\, w_Q(w)\, dx, \tag{39}$$

$$\frac{n}{c_{n-1/2}(w_Q)} = \int_{-\infty}^{\infty} p_n(w_Q; x)\, p_{n-1}(w_Q; x)\, Q'(x)\, w_Q(x)\, dx, \tag{40}$$

and

$$\int_{-\infty}^{\infty} p_n^2(w_Q; x)\, x Q'(x)\, w_Q(x)\, dx = \frac{2n+1}{2}. \tag{41}$$

THEOREM 9. *Let w_Q be defined by* (36) *where $Q(x)$ is even, differentiable for $x > 0$ and $Q'(x)$ is increasing then*

$$\tfrac{1}{2} q_n \leqslant c_{n-1/2}(w_Q) \leqslant 2 q_n \tag{42}$$

and

$$\tfrac{1}{2} q_{n-1} \leqslant X_n(w_Q) \qquad \leqslant 4 q_{n-1}. \tag{43}$$

Proof. By (39) and (41) we have

$$\begin{aligned}
c_{n-1/2}(w_Q) \leqslant q_n &\int_{-q_n}^{q_n} |p_n(w_Q; x)|\, |p_{n-1}(w_Q; x)|\, w_Q(x)\, dx \\
&+ \frac{1}{Q'(q_n)} \int_{|x| > q_n} |p_n(w_Q; x)|\, |p_{n-1}(w_Q; x)|\, x Q'(x)\, w_Q(x)\, dx \\
&\leqslant q_n \left\{ \int_{-\infty}^{\infty} p_n^2(w_Q; x)\, w_Q(x)\, dx \int_{-\infty}^{\infty} p_{n-1}^2(w_Q; x)\, w_Q(x)\, dx \right\}^{1/2} \\
&+ \frac{1}{Q'(q_n)} \left\{ \int_{-\infty}^{\infty} p_n^2(w_Q; x)\, x Q'(x)\, w_Q(x)\, dx \right. \\
&\times \left. \int_{-\infty}^{\infty} p_{n-1}^2(w_Q; x)\, x Q'(x)\, w_Q(x)\, dx \right\}^{1/2} \\
&= q_n + \frac{1}{Q'(q_n)} \frac{\sqrt{2n+1}}{2} \frac{\sqrt{2n-1}}{2} = q_n + \frac{q_n}{n} \sqrt{\frac{4n^2-1}{4}} \leqslant 2 q_n
\end{aligned}$$

and similarly by (40) and (41)

$$\frac{n}{c_{n-1/2}(w_Q)} \leqslant Q'(q_n) \int_{-q_n}^{q_n} |p_n(w_Q; x)|\, |p_{n-1}(w_Q; x)|\, w_Q(x)\, dx$$

$$+\frac{1}{q_n} \int_{|x|>q_n} |p_n(w_Q; x)|\, |p_{n-1}(w_Q; x)|\, xQ'(x)\, w_Q(x)\, dx$$

$$\leqslant Q'(q_n) + \frac{1}{q_n} \sqrt{\frac{2n-1}{2}} \sqrt{\frac{2n+1}{2}} \leqslant \frac{2n}{q_n},$$

that is, $c_{n-1/2}(w_Q) \geqslant \frac{1}{2} q_n$. Equation (43) follows from (12), (13), and (42).

Note added in proof. Regarding recent improvements of the results of this paper, including a partial resolution of Freud's conjecture in the sentence following (35), see "Géza Freud, Orthogonal Polynomials and Christoffel Functions," by Paul Nevai in this volume of *J. Approx. Theory.*

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